Elementary matrices

Definition: An $n \times n$ matrix is an <u>elementary matrix</u> if it can be obtained from I_n by a single elementary row operation. It is of type I, II, or III if the operation is that type.

swapping rows | nonzero multiple of row | row + multiple of another row

Ex:

switch row 1 and row 2 | 2 times row 1 | row 2 + −5 times row 1

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

are elementary matrices, but

$$\begin{bmatrix} -1 & 5 \\ 0 & 1 \end{bmatrix}$$

is not.

What happens if we multiply by an elementary matrix?

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$E_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} a & b & c \\ d-5a & e-5b & f-5c \end{bmatrix}$$

i.e. each product is A after the corresponding row operation is applied. This holds in general:

Theorem: If A is an $m \times n$ matrix and E is an elementary $m \times m$ matrix, then EA is the matrix obtained by performing the row operation corresponding to E on A.

Every row operation can be "undone" by performing an "inverse" operation of the same type.

Ex: $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{to row 2}]{\text{add } 3\cdot\text{row 1}} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \xrightarrow[\text{from row 2}]{\text{subtract } 3\cdot\text{row 1}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

More generally, we can reverse each type of row operation as follows:

	Operation	Inverse operation
Type I:	Swap rows p and q	Swap rows p and q
Type II:	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
Type III:	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$.

This gives us a way to construct inverses for elementary matrices:

That is, if E is an elementary matrix, and F is the elementary matrix corresponding to the inverse operation, then $FE = I$, so $F = E^{-1}$.

Ex: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -17 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 17 & 0 & 1 \end{bmatrix}$$

As a result, the following holds:

All elementary matrices are invertible

Inverses and elementary matrices

Recall that a matrix A is invertible if and only if $A \to I$ by elementary row operations. Thus, we can also get I by multiplying A by elementary matrices.

Ex: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$[A \mid I] \xrightarrow[\text{①}\leftrightarrow\text{②}]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{①}+2\cdot\text{②}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}\text{③}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/5 \end{matrix}}_{\text{"}A^{-1}\text{"}} \\ 0 & 0 & 1 & \end{array}\right]$$

Corresponding elementary matrices: $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$

so $E_3E_2E_1A = I$ and $E_3E_2E_1 I = A^{-1}$

$$\Rightarrow \quad A^{-1} = E_3E_2E_1 \quad \text{and} \quad A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}}_{\substack{\text{also elementary} \\ \text{matrices.}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Theorem: A square matrix is invertible if and only if it is the product of elementary matrices.

Moreover, for any two matrices A, B such that $A \longrightarrow B$ via row operations, we have $B = UA$ for U an invertible square matrix (the product of the corresponding elementary matrices).

Equivalently, we can start with

$$\begin{bmatrix} A & I_m \end{bmatrix} \xrightarrow{\text{and get}} \begin{bmatrix} \underset{UA}{\overset{\shortparallel}{B}} & U \end{bmatrix}$$

Practice problems: 2.5 : 1ace, 2def, 3, 6bc, 8bd, 12c